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A Two-Echelon Inventory Optimization Model with Demand Time Window Considerations

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Abstract. This paper studies a two-echelon dynamic lot-sizing model with demand time windows and early and late delivery penalties. The problem is motivated by third-party logistics and vendor managed inventory applications in the computer industry where delivery time windows are typically specified under a time definite delivery contract. Studying the optimality properties of the problem, the paper provides polynomial time algorithms that require $O(T^3)$ computational complexity if backlogging is not allowed and $O(T^5)$ computational complexity if backlogging is allowed.

Key words: Demand time-window, Dynamic programming, Lot-sizing

1. Introduction

Recently, time definite delivery agreements have become a popular component of supply contracts in both third-party logistics (TPL) and vendor managed inventory (VMI) practices in the electronics industry in Texas. Under a typical time definite delivery agreement in the computer industry, a TPL provider is in charge of the outbound distribution and VMI programs of the manufacturer. The inventory and demand information of the downstream supply chain member (e.g., a distribution center or retailer) is accessible to the supplier (e.g., a TPL provider). After reviewing the downstream inventory levels, the TPL provider is empowered to make decisions regarding the quantity/timing of re-supply (Çetinkaya and Lee, 2000). The distribution center (DC), however, requests timely deliveries by imposing maximum holding times (demand time windows) for shipments. Naturally, such a system is favorable for effective VMI where the supplier is responsible for managing inventories at the downstream supply chain member and guaranteeing timely delivery by satisfying the demand time window constraints.

In the particular application of interest, finished products are shipped from the manufacturer to a third-party warehouse (TPW) for temporary storage and distribution. A linear inventory carrying is incurred for each unit held in inventory at

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the TPW per unit time, and a fixed set-up cost is incurred each time the stock is replenished at the TPW. Products are then delivered from the TPW to a DC in bulk replenishment quantities. For each dispatch to the DC from the TPW, again, a fixed set-up cost is incurred. We consider the case where each demand at the DC has a time window, specified under the terms of a contract between the TPW and DC, during which the demand can be satisfied without penalty. If a demand is delivered prior to its earliest delivery time, a linear pre-shipping penalty (holding cost at the DC level) is incurred for each unit per unit time until the earliest delivery time is reached. On the other hand, if a demand is delivered later than its latest delivery time, then a linear waiting cost is incurred for each unit per unit time until the actual delivery time.

The problem is motivated by a real-life application in the computer industry, and it is treated as a two-echelon dynamic lot-sizing model with demand time window considerations. The objective is to find an integrated replenishment policy for the TPW and DC simultaneously to satisfy all demands at the DC at minimum cost. As for the traditional dynamic lot-sizing literature, this paper assumes that demand is known in advance. This class of models has a wide domain of applications where orders have been placed in advance or contracts have been signed ahead of time specifying deliveries for the next few periods.

In a recent paper, Lee et al. (2001) generalize the classical dynamic lot-sizing model to consider demand time windows, and they provide a polynomial time algorithm for both cases — where backorders are allowed and where they are not. Here, we extend this recent paper by considering a two-echelon problem arising in a TPL application. Studying the optimality properties of the problem, the paper provides polynomial time algorithms that require $O(T^3)$ computational complexity if backlogging is not allowed and $O(T^5)$ computational complexity if backlogging is allowed. The solution procedure is based on the dynamic programming approach.

The remainder of this paper is organized as follows. A summary of the relevant literature is presented in Section 2. In Section 3, we introduce the notation and discuss some structural properties of the problem. The case in which backlogging is allowed is considered in Section 4. In Section 5, we consider the case in which backlogging is not allowed. Finally, Section 6 presents a discussion of future work and concludes the paper.

2. Relevant literature

Following Wagner and Whitin's (1958) seminal work, a number of highly influential papers studied the dynamic lot-sizing problem and its applications. See, for example, Zangwill (1966), Florian and Klein (1971), Jagannathan and Rao (1973), Love (1973), Swoveland (1975), Bitran and Yanasse (1982), Bitran et al. (1984), Lee and Denardo (1986), Chung and Lin (1988), Federgruen and Tzur (1991), Wagelmans et al. (1992), Aggarwal and Park (1993), Chen et al. (1994), and Shaw and Wagelmans (1998). Although the entire literature on dynamic lot-sizing is closely related to our study, a comprehensive review of this popular research area is beyond the scope of this paper. Hence, we focus on some of the fundamental papers for the purpose of providing a brief overview.

In a pioneering paper, Zangwill (1969) considered the multi-echelon dynamic lot-sizing model for a single item, and he presented a network approach for the problem. Other papers that analyzed multi-echelon problems for a single item include Crowston and Wagner (1973), Blackburn and Millen (1982), and Diaby and Martel (1993), and Lee et al. (2003). These papers extended the multi-echelon lot-sizing literature to consider general cost structures and capacity constraints. The particular problem of interest in the current paper also extends the multi-echelon lot-sizing literature to consider demand time window constraints.

It is also worth noting that a number of existing papers analyzed machine scheduling and vehicle routing problems with demand time windows. For machine scheduling problems, see Cheng (1988), Kramer and Lee (1993), Liman and Ramaswamy (1994), Weng and Ventura (1994), and Liman et al. (1996); and for vehicle routing problems, see Dumas et al. (1991), Desrochers et al. (1992), Bramel and Simchi-Levi (1997), Kohl and Madsen (1997), and Fisher et al. (1997). However, the concept of a demand time window is relatively new in the context of inventory control.

As we have already mentioned, Lee et al. (2001) analyzed the single-echelon version of the problem discussed here, and they presented an $O(T^2)$ algorithm for the case in which backlogging is not allowed. They also showed that, if backlogging is allowed, then the single-echelon problem under demand time windows could be solved in $O(T^3)$ time. The two-echelon problem studied here is naturally more complicated. However, the problem can still be solved in polynomial time. The optimal algorithms developed here require $O(T^3)$ computational complexity for the case where backlogging is not allowed and $O(T^5)$ computational complexity for the case where it is allowed.

3. Notation and optimality properties

It is important to note that for the problem of interest, demands need to be differentiated only if they have distinct time windows. Let N denote the number of demands throughout the planning horizon of T time periods. Observe that, $N \leqslant (T^2+T)/2$. For each $n \in \{1,\ldots,N\}$, let d_n represent the quantity of demand n; and E_n and L_n denote the corresponding earliest and latest delivery times, respectively where $1 \leqslant E_n \leqslant L_n \leqslant T$ for all n. For $1 \leqslant t \leqslant T$,

- K_t denotes the fixed cost of a replenishment at the TPW in period t,
- S_t denotes the fixed cost of a dispatch to the DC in period t,
- p_t denotes the unit procurement cost in period t,

- h'_t denotes the unit holding cost in period t at the TPW,
- h_t denotes the unit pre-shipping penalty (holding cost) in period t at the DC,
- w_t denote the unit late shipping (waiting/backlogging penalty) in period t at the DC,
- x_t denotes the replenishment quantity at the TPW in period t,
- y_t denotes the dispatch quantity, i.e., the amount dispatched to the DC, in period t,
- d_{nt} denotes the amount of d_n that is scheduled to be satisfied in period t,
- I'_t denotes the on-hand inventory level at the TPW at the end of period t,
- I_t^+ denotes the on-hand inventory level at the DC at the end of period t, and
- I_t^- denotes the quantity backordered at the DC at the end of period t.
- For $s \leqslant t$, we define

$$h'(s,t) = h'_s + \cdots + h'_t, h(s,t) = h_s + \cdots + h_t, \text{ and}$$
$$w(s,t) = w_s + \cdots + w_t.$$

- For $s > t$ we define $h'(s,t) = 0$, $h(s,t) = 0$, and $w(s,t) = 0$.

Using this notation, the two-echelon dynamic lot-sizing problem with demand time windows can be formulated as an integer program as follows:

$$\text{Min} \sum_{t=1}^{T} (K_t \cdot \delta(x_t) + p_t \cdot x_t + S_t \cdot \delta(y_t) + h'_t \cdot I'_t + h_t \cdot I_t^+ + w_t \cdot I_t^-) \tag{1}$$

Subject to

$$I'_{t-1} + x_t - y_t = I'_t, \qquad t = 1, \ldots, T, \tag{2}$$

$$I_{t-1}^+ - I_{t-1}^- + y_t - \sum_{n=1}^{N} d_{nt} = I_t^+ - I_t^-, \qquad t = 1, \ldots, T, \tag{3}$$

$$\sum_{t=E_n}^{L_n} d_{nt} = d_n, \qquad n = 1, \ldots, N, \tag{4}$$

$$d_{nt} \geqslant 0, \qquad n = 1, \ldots, N, t = E_n, \ldots, L_n, \tag{5}$$

$$d_{nt} = 0, \qquad n = 1, \ldots, N, t = 1, \ldots, E_n - 1, \tag{6}$$

$$d_{nt} = 0, \qquad n = 1, \ldots, N, t = L_n + 1, \ldots, T, \tag{7}$$

$$x_t \geqslant 0, y_t \geqslant 0, I'_t \geqslant 0, I_t^+, \geqslant 0, I_t^- \geqslant 0, \qquad t = 1, \ldots, T, \tag{8}$$

$$I_0^+ = I_0^- = I'_0, = 0, \tag{9}$$

where $\delta(a) = 1$ if $a > 0$ and 0 otherwise.

For notational convenience, we also define $I_t = I_t^+ - I_t^-$. Here, I_t represents the inventory level at the DC at the end of period t. It can be shown easily that in any optimal solution, $I_t^+ \cdot I_t^- = 0$. Also, note that I_t can be negative while I_t' is non-negative.

Following a common assumption of the multi-echelon inventory literature, we assume that $\boldsymbol{h_t' \leqslant h_t}$ **for all** $\boldsymbol{t}$. This assumption is easily justifiable for all practical purposes since inventory movement/transportation from upper echelons to lower echelons is a value-added operation. Here, we consider the case where there are **no speculative motives** for holding inventory or backordering so $\boldsymbol{K_t \geqslant K_{t+1}, p_t \geqslant p_{t+1}}$ and $\boldsymbol{p_t + w_{t-1} \geqslant p_{t-1}}$ for all $\boldsymbol{t}$. That is, the fixed ordering (setup) costs and per-unit procurement (variable production) costs are non-increasing which is the case, for example, with technology products where there is a "learning" effect in production. The dispatch costs, S_t, can take arbitrary values. Under these assumptions, the following properties hold in general, whether backlogging is allowed or not, and they can be proven in a straightforward manner. It is worth noting that the assumptions $K_t \geqslant K_{t+1}$ and $p_t \geqslant p_{t+1}$ are useful for proving Property 2. The assumption postulating that $p_t \geqslant p_{t+1}$ is also useful for proving Lemma 1. Finally, the assumptions postulating that $p_t \geqslant p_{t+1}$ and $p_t + w_{t-1} \geqslant p_{t-1}$ are needed for proving Lemma 3. Without Property 1 and Lemmas 1 and 3, a polynomial time solution does not seem possible.

PROPERTY 1. There exists an optimal solution such that $I_{t-1}' \cdot x_t = 0$ for all $t = 1, \ldots, T$, i.e., if an inbound replenishment exists at period t $(x_t > 0)$ then $I_{t-1}' = 0$.

PROPERTY 2. An inbound replenishment is received only when an outbound dispatch is made, i.e., for a given $t, x_t > 0$ only if $y_t > 0$.

PROPERTY 3. There exists an optimal solution such that for each $k = 1, \ldots, T$, either $x_k = y_k + y_{k+1} + \cdots + y_l$ for some $l \geqslant k$, or $x_k = 0$.

PROPERTY 4. There exists an optimal solution in which demand is not split. That is, there exists an optimal policy such that, for each demand, the entire quantity is satisfied by the same dispatch.

For the problem with traditional demand, it is optimal to satisfy demands on a first-come first-served basis. However, for the problem with demand time windows, it is not necessarily optimal to satisfy the demands in the same fashion (see Lee et al., 2001). Hence, the optimal policy should specify the following simultaneously:

(i) *The replenishment plan* specifies "when, and in what quantities, to replenish the stock at the TPW."
(ii) *The dispatch plan* specifies "when, and in what quantities, to release an outbound shipment to the DC, and in which order to satisfy the demands."

In the remainder of this paper, period t is called **a replenishment period** if $x_t > 0$, and it is called a **dispatch period** if $y_t > 0$. Observe that a replenishment period is always a dispatch period, and, as we have already discussed, each dispatch is supplied solely by a single replenishment. Before presenting a detailed analysis of additional properties of the problem, let us introduce more notations.

- For $u \leqslant v$,
  - $D(u,v)$ denotes the sum of all d_n with $E_n = u$ and $L_n = v$,
  - $R(u,v)$ denotes the sum of all d_n with $u \leqslant L_n \leqslant v$, and
  - $W(u,v)$ denotes the sum of all d_n with $u \leqslant E_n \leqslant L_n \leqslant v$.
- For $u > v$, we define $D(u,v) = 0$, $R(u,v) = 0$ and $W(u,v) = 0$.
- For each triple (k,v,t) such that $k \leqslant v \leqslant t$, $A(k,v,t)$ denotes the sum of all d_n with $E_n = k$ and $v \leqslant L_n \leqslant t$, i.e.,

$$A(k,v,t) = \sum_{j=v}^{t} D(k,j), \text{ and}$$

$G(k,v,t)$ denotes the sum of all d_n with $k \leqslant E_n \leqslant v$ and $v \leqslant L_n \leqslant t$, i.e.,

$$G(k,v,t) = \sum_{i=k}^{v} A(i,v,t).$$

- Finally, for $v > t$ or $k > v$, we define $A(k,v,t) = 0$ and $G(k,v,t) = 0$.

It can be easily shown that it takes $O(T^2)$ time to find the values of $D(u,v)$, $R(u,v)$ and $W(u,v)$ for all $1 \leqslant u \leqslant v \leqslant T$. Also note that the values of $A(k,v,t)$ and $G(k,v,t)$ for all $1 \leqslant k \leqslant v \leqslant t \leqslant T$ can be found in $O(T^3)$ time.

We consider the case in which backlogging is not allowed in Section 4 and the case in which backlogging is allowed in Section 5.

4. Backlogging is not allowed

In order to develop an efficient dynamic programming based algorithm, we first present an important structural property that permits us to decompose the original problem into a sequence of smaller problems.

LEMMA 1. *There exists an optimal solution such that if i and $j+1$ are consecutive replenishment periods, then all d_n with $i \leqslant L_n \leqslant j$ are supplied by the replenishment in period i.*

Proof. Suppose d_n with $i \leqslant L_n \leqslant j$ is not supplied by the replenishment in period i in an optimal solution. Since backlogging is not allowed, d_n must be supplied by a replenishment in some period $k < i$. Let $g, k \leqslant g < i$, denote the dispatch period that satisfies d_n. Consider a new solution which is the same as before, except d_n is satisfied by the replenishment and the dispatch in period i. Clearly, the TPW inventory level at the end of any period is not increased. It can be shown that the incremental cost is less than or equal to $p_i + h(i, E_n - 1) - p_k - h'(k, g-1) - h(g, E_n - 1) \leqslant 0$, since $p_i \leqslant p_k, h(i, E_n - 1) \leqslant h(g, E_n - 1)$ and $h'(k, g-1) \geqslant 0$. Thus, this perturbation will not increase the total cost. Hence, the new solution is also optimal. □

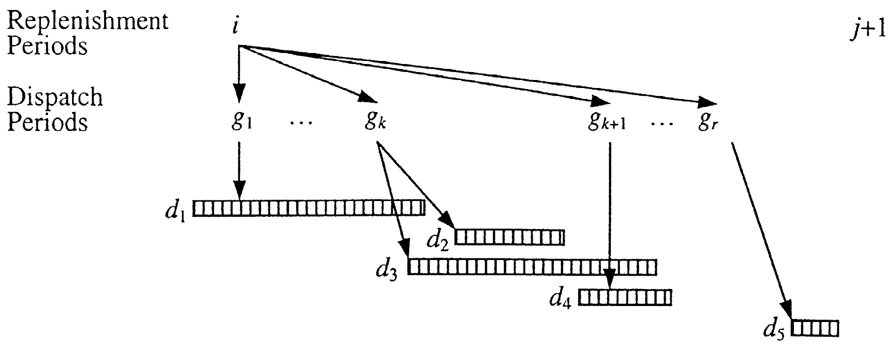

Figure 1. Optimal policy for finding the value of $C(i,j)$.

For $1 \leqslant i \leqslant j \leqslant T$, we define $\boldsymbol{C(i,j)}$ **as the minimum total cost of replenishing in period** $\boldsymbol{i}$ **to satisfy all** $\boldsymbol{d_n}$ **with** $\boldsymbol{i \leqslant L_n \leqslant j}$ including the set-up cost of the replenishment in period i, the procurement cost, the set-up costs of the dispatches supplied by the replenishment in period i, the TPW holding costs, and the DC pre-shipping penalties. For $1 \leqslant t \leqslant T$, let $\boldsymbol{F(t)}$ **denote the minimum total cost of satisfying those** $\boldsymbol{d_n}$ with $\boldsymbol{L_n \leqslant t}$. Using Lemma 1 and the principle of optimality, the optimal solution can be computed using the following recursive equations:

$$F(0) = 0,$$

$$F(j) = \min\{F(i-1)+C(i,j) : 1 \leqslant i \leqslant j\}, \text{ for } j=1,\ldots,T \tag{10}$$

Thus, **the optimal cost is** $\boldsymbol{F(T)}$. If the values of $C(i,j)$ for all $1 \leqslant i \leqslant j \leqslant T$ have been computed then it takes $O(T^2)$ time to find the optimal value. Thus, the main concern is how to compute $C(i,j)$ in an efficient way. The following lemma is useful for computing the value of $C(i,j)$.

LEMMA 2. *There exists an optimal solution such that if* $g_1 < g_2 < \cdots < g_r$ *are successive dispatch periods supplied by the replenishment in period i, where* $g_1 = i$ *and* $g_r \leqslant j$*, then the following observations are true (see Figure 1).*

(*i*) *If* $E_n \leqslant g_1 \leqslant L_n \leqslant j$*, then* d_n *is satisfied by the dispatch in period* $g_1 = i$.

(*ii*) *If* $g_k < E_n \leqslant L_n < g_{k+1}$*, for some* k*,* $1 \leqslant k < r$*, then* d_n *is satisfied by the dispatch in period* g_k.

(*iii*) *If* $g_k < E_n \leqslant g_{k+1} \leqslant L_n \leqslant j$*, for some* k*,* $1 \leqslant k < r$*, then* d_n *is satisfied by*

- *the dispatch in period* g_k*, if* $h(g_k, E_n - 1) \leqslant h'(g_k, g_{k+1} - 1)$*,*
- *the dispatch in period* g_{k+1}*, if* $h(g_k, E_n - 1) > h'(g_k, g_{k+1} - 1)$.

(*iv*) *If* $g_r < E_n \leqslant L_n \leqslant j$*, then* d_n *is satisfied by the dispatch in period* g_r.

Proof. The proof is straightforward; thus, it is omitted. □

Consider the illustration in Figure 1 where each bar represents the corresponding time window of a demand, e.g., $g_k < E_2 \leqslant L_2 < g_{k+1}$, and dispatch quantities in periods $g_1, \ldots, g_r$ are supplied by the replenishment in period i. Under the optimal dispatch schedule, d_1 is satisfied by the dispatch in period g_1, d_2 and d_3 are satisfied by the dispatch in period g_k, d_4 is satisfied by the dispatch in period g_{k+1}, and d_5 is satisfied by the dispatch in period g_r. Note that $h(g_k, E_3-1) \leqslant h'(g_k, g_{k+1}-1)$ and $h(g_k, E_4-1) > h'(g_k, g_{k+1}-1)$.

Observe that any demand d_n in the problem for finding $C(i,j)$ can be classified as either case i) $E_n \leqslant i \leqslant L_n$ or case ii) $i < E_n \leqslant L_n \leqslant j$. Thus, those demands in case i) are satisfied by the dispatch in period i so that no holding cost or pre-shipping penalty is incurred. Those demands in case ii) are subject to holding cost and/or pre-shipping penalty. For $i<j$, we define $M(i,j)$ as the minimum cost of satisfying all d_n with $i < E_n \leqslant L_n \leqslant j$, including the costs of all dispatches to the DC supplied by the replenishment in period i (i is always a dispatch period) and the corresponding *holding costs and pre-shipping penalties*. If $i=j$, then we define $M(i,j)=S_j$.

The value of $C(i,j)$, $i \leqslant j$, is thus given by

$$C(i,j)=\begin{cases} 0 & \text{if } R(i,j)=0 \\ K_i + p_i \cdot R(i,j) + M(i,j) & \text{otherwise} \end{cases} \tag{11}$$

Note that $K_i + p_i \cdot R(i,j)$ is the cost of replenishing $R(i,j)$ in period i. Hence, it is important to develop an efficient way to compute the values of $M(i,j)$ for all $1 \leqslant i \leqslant j \leqslant T$. Define

- $V(u,v,j)$, $1 \leqslant u < v \leqslant j \leqslant T$, is the minimum cost associated with the *holding costs* and the *pre-shipping penalties* to satisfy all d_n with $u < E_n \leqslant v$ and $L_n \leqslant j$ (totally $W(u+1,j) - W(v+1,j)$ units) by the dispatches in periods u and v, given that those amounts are available at the beginning of period u. Note that we intentionally do not include the set-up costs of the dispatches in periods u and v.
- $V'(u,j,j), 1 \leqslant u < j \leqslant T$, is the minimum cost associated with the *pre-shipping penalties* to satisfy all d_n with $u < E_n \leqslant L_n \leqslant j$ (totally $W(u+1,j)$ units) by the dispatch in period u given that those amounts are available at the beginning of period u. Note that we intentionally do not include the set-up cost of the dispatch in period u.

For $1 \leqslant i \leqslant j \leqslant T$, the values of $M(i,j)$ can be obtained using the following recursive equations.

For $j=1,\ldots,T$,

$$M(j,j)=S_j,$$

and for $i=j-1, j-2, \ldots, 1$,

$$M(i,j)=S_i+\min\left\{\begin{array}{l} V'(i,j,j) \\ \min\{M(v,j)+h'(i,v-1)\cdot W(v+1,j) \\ \qquad +V(i,v,j): i<v\leqslant j\} \end{array}\right\} \tag{12}$$

Note that $W(v+1,j)$ is the quantity considered in computing $M(v,j)$.

If the values of $V(u,v,j)$ for all $1 \leqslant u < v \leqslant j \leqslant T$ and the value of $V'(u,j,j)$ for all $1 \leqslant u < j \leqslant T$ have been computed, then the values of $M(i,j)$

for all $1 \leqslant i \leqslant j \leqslant T$ can be found in $O(T^3)$ time. Hence, we need to find an efficient way to compute all values of $V(u,v,j)$ and $V'(u,j,j)$.

4.1. COMPUTING $V(u,j,j)$

By Lemma 2 (case iv), all d_n with $u < E_n \leqslant L_n \leqslant j$ are satisfied by the dispatch in period u. Observe that $V'(u,j,j) = \sum_{k=u}^{j-1} h_k \cdot W(k+1,j)$. Hence, the value of $V'(u,j,j)$, $1 \leqslant u < j \leqslant T$, can be obtained using the following recursive equations.

For $j = 2, \ldots, T$, let

$$V'(j,j,j) = 0,$$

and for $u = j-1, j-2, \ldots, 1$,

$$V'(u,j,j) = V'(u+1,j,j) + h_u \cdot W(u+1,j). \tag{13}$$

Clearly, it takes $O(T^2)$ time to compute the values of $V'(u,j,j)$ for all $1 \leqslant u < j \leqslant T$.

4.2. COMPUTING $V(u,v,j)$

We need to consider assigning d_n with $u < E_n \leqslant v$ and $L_n \leqslant j$ to either the dispatch in period u or the dispatch in period v. These demands can be separated into two groups:

(i) d_n with $u < E_n \leqslant L_n < v \leqslant j$ (case (ii) in Lemma 2), and

(ii) d_n with $u < E_n \leqslant v \leqslant L_n \leqslant j$ (case (iii) in Lemma 2).

Since backlogging is not allowed, all d_n with $u < E_n \leqslant L_n < v$ must be satisfied by the dispatch in period u; hence, the corresponding pre-shipping penalties can be calculated in a straightforward manner. For d_n with $u < E_n \leqslant v \leqslant L_n \leqslant j$, if $h(u, E_n - 1) \leqslant h'(u, v-1)$, then d_n is satisfied by the dispatch in period u; otherwise, d_n is satisfied by the dispatch in period v. For $1 \leqslant u < v \leqslant T$, define

$$\Delta(u,v) = \max\{k : h(u,k-1) \leqslant h'(u,v-1)\}. \tag{14}$$

Thus, if $E_n \leqslant \Delta(u,v)$, then d_n is satisfied by the dispatch in period u. Otherwise, if $E_n > \Delta(u,v)$, then d_n is satisfied by the dispatch in period v. Clearly, we have $u \leqslant \Delta(u,v) \leqslant v$ and $\Delta(u,v) \leqslant \Delta(u+1,v)$; thus, it can be easily shown that all values of $\Delta(u,v)$ can be obtained in $O(T^2)$ time.

Suppose the value of $V(u+1,v,j)$ is known for some $1\leqslant u<v\leqslant j\leqslant T$. The value of $V(u,v,j)$ can be computed by using the following equation.

$$V(u,v,j) = V(u+1,v,j)+h_u\cdot W(u+1,v-1)+h_u\cdot \sum_{k=u+1}^{\Delta(u,v)} A(k,v,j)$$
$$+\sum_{k=\Delta(u,v)+1}^{\Delta(u+1,v)} [(h'(u,v-1)-h(u+1,k-1))\cdot A(k,v,j)]$$
$$+h'_u\cdot \sum_{k=\Delta(u+1,v)+1}^{v} A(k,v,j). \tag{15}$$

The justification of Equation (15) follows. When u and v become two consecutive dispatch periods instead of $u+1$ and v, the incremental cost of $V(u,v,j)-V(u+1,v,j)$ is incurred. The following provides detailed discussion on the incremental cost.

- All d_n with $u<E_n\leqslant L_n<v\leqslant j$ (case (ii) in Lemma 2) are satisfied by the dispatch in period u, and the incremental pre-shipping penalty is $h_u\cdot W(u+1,v-1)$.
- When $u+1$ and v are consecutive dispatch periods, demands corresponding to $A(k,v,j)$ for $k=u+2,\ldots,\Delta(u+1,v)$ are satisfied by the dispatch in period $u+1$; demands corresponding to $A(k,v,j)$ for $k=\Delta(u+1,v)+1,\ldots,v$ are satisfied by the dispatch in period v. Similarly, if u and v are consecutive dispatch periods, then demands corresponding to $A(k,v,j)$ for $k=u+1,\ldots,\Delta(u,v)$ should be satisfied by the dispatch in period u; demands corresponding to $A(k,v,j)$ for $k=\Delta(u,v)+1,\ldots,v$ should be satisfied by the dispatch in period v. Thus, the incremental cost for those d_n with $u<E_n\leqslant v\leqslant L_n\leqslant j$ is the sum of the following three items:

  i) The incremental pre-shipping penalty for demands corresponding to $A(k,v,j)$ for $k=u+1,\ldots,\Delta(u,v)$ which is given by
  $$h_u\cdot \sum_{k=u+1}^{\Delta(u,v)} A(k,v,j).$$
  ii) The incremental cost of satisfying demands corresponding to $A(k,v,j)$ for $k=\Delta(u,v)+1,\ldots,\Delta(u+1,v)$ by the dispatch in period v which is given by
  $$\sum_{k=\Delta(u,v)+1}^{\Delta(u+1,v)} [(h'(u,v-1)-h(u+1,k-1))\cdot A(k,v,j)].$$
  (These demands were satisfied by the dispatch in period $u+1$ when $u+1$ and v were consecutive dispatch periods.)

iii) The incremental holding cost for demands corresponding to $A(k,v,j)$ for $k=\Delta(u+1,v)+1,\ldots,v$ which is given by

$$h'_u \cdot \sum_{k=\Delta(u+1,v)+1}^{v} A(k,v,j).$$

Using Equation (15), the values of $V(u,v,j)$ for all $1 \leqslant u < v \leqslant j \leqslant T$ can be computed in $O(T^4)$ time. However, the equation can be rewritten as follows. For given v and j, such that $1 < v \leqslant j \leqslant T$, let

$$V(v,v,j)=0,$$

and for $u=v-1, v-2,\ldots,1$,

$$\begin{aligned} V(u,v,j) = {} & V(u+1,v,j)+h_u \cdot W(u+1,v-1)+h_u \cdot [G(u+1,v,j) \\ & -G(\Delta(u,v)+1,v,j)]+\sum_{k=\Delta(u,v)+1}^{\Delta(u+1,v)} [(h'(u,v-1)-h(u+1,k-1)) \\ & \cdot A(k,v,j)]+h'_u \cdot G(\Delta(u+1,v)+1,v,j). \end{aligned} \tag{16}$$

For given v and $j, 1 < v \leqslant j \leqslant T$, using Equation (16), the values of $V(u,v,j)$ for all $u < v$ can be computed in $O(v)$ time. This is because, by definition, $1 \leqslant u \leqslant \Delta(u,v) \leqslant \Delta(u+1,v) \leqslant v \leqslant T$. Thus, the values of $V(u,v,j)$ for all $1 \leqslant u < v \leqslant j \leqslant T$ can be computed in $O(T^3)$ time.

Recall that it takes $O(T^2)$ and $O(T^3)$ time to find all values of $V'(u,j,j)$ and $V(u,v,j)$, respectively. It takes $O(T^3)$ time to compute the values of $M(i,j)$ for all $1 \leqslant i \leqslant j \leqslant T$. Finding the values of $C(i,j)$ for all $1 \leqslant i \leqslant j \leqslant T$ takes $O(T^2)$ time. Finally, the optimal cost, $F(T)$, can be obtained in $O(T^2)$. Hence, the computational complexity of the case in which backlogging is not allowed is $O(T^3)$.

5. Backlogging is allowed

For the case where backlogging is allowed, we can simplify the explanation of the proposed algorithm by making a simple assumption. That is, without loss of generality, we let $\boldsymbol{K_0 = K_{T+1} = S_0 = S_{T+1} = 0}$, **and** $\boldsymbol{p_0 = w_T = \infty}$, and we argue that the optimal solution obtained with this cost structure is identical to the optimal solution of the problem of interest. Note that the "no speculative motives" assumption still holds since $p_t \geqslant p_{t+1}$ and $p_t + w_{t-1} \geqslant p_{t-1}$ for all $t=0,\ldots,T+1$. Thus, we can safely assume that there is an optimal solution in which periods 0 *and* $T+1$ *are both replenishment and dispatch periods*. Although we force period 0 to be a replenishment period, the corresponding replenishment quantity has to be zero due to the high procurement cost. If there is no replenishment in period 0, then there is no dispatch due to lack of inventory. Also, the dispatch quantity in period $T+1$ has to be zero due to the high backlogging cost. Hence, the replenishment quantity in period $T+1$ must also be zero. As a result, all demands must be satisfied by replenishments and dispatches during periods $1, \ldots, T$. It follows that assuming "$\boldsymbol{K_0 = K_{T+1} = S_0 = S_{T+1} = 0}$," **and** "$\boldsymbol{p_0 = w_T = \infty}$" do not alter the optimal solution of the problem.

The following lemma and property are useful for developing an optimal dynamic programming algorithm for the case where backlogging is allowed.

LEMMA 3. *There exists an optimal solution such that if i and j are consecutive replenishment periods, then all d_n with $i \leqslant L_n \leqslant j$ are supplied by the replenishment in either period i or j.*

Proof. Suppose d_n with $i \leqslant L_n \leqslant j$ is not supplied by the replenishment in period i in an optimal solution. Consider the following two cases:

CASE 1. d_n is supplied by a replenishment in some period $k < i$

Let $g, k \leqslant g < i$, denote the dispatch period that satisfies d_n. Consider a new solution which is the same as before, except d_n is satisfied by the replenishment and the dispatch in period i. Clearly, the TPW inventory level at the end of any period is not increased. It can be shown that the incremental cost is less than or equal to $p_i + h(i, E_n - 1) - p_k - h'(k, g-1) - h(g, E_n - 1) \leqslant 0$, since $p_i \leqslant p_k, h(i, E_n - 1) \leqslant h(g, E_n - 1)$ and $h'(k, g-1) \geqslant 0$.

CASE 2. d_n is supplied by a replenishment in some period $k > j$

Let $g, k \leqslant g$, denote the dispatch period that satisfies d_n. Consider a new solution that is the same as before except d_n is satisfied by the replenishment and the dispatch in period j. Clearly, the TPW inventory level at the end of any period is not increased. It can be shown that the incremental cost is less than or equal to $p_j + w(L_n, j-1) - p_k - h'(k, g-1) - w(L_n, g-1) = p_j - p_k - w(j, k-1) - w(k, g-1) - h'(k, g-1) \leqslant 0$, since $p_j \leqslant p_k + w(j, g-1), w(k, g-1) \geqslant 0$ and $h'(k, g-1) \geqslant 0$.

Thus, the above perturbation schemes do not increase the total cost. Hence, the new solution is also optimal. □

PROPERTY 5. There exists an optimal solution such that, if d_n is supplied by the replenishment in period j such that $L_n \leqslant j$, then d_n must also be satisfied by the dispatch in period j.

Proof. Since period j is a replenishment period, it is always a dispatch period. Among all dispatches supplied by the replenishment in period j, it is obvious that the least waiting cost is incurred when d_n is satisfied by a dispatch in period j. □

For $0 \leqslant i < j \leqslant T+1$, we define $\boldsymbol{P(i,j)}$ **as the minimum total cost of replenishing in periods $\boldsymbol{i}$ and $\boldsymbol{j}$ to satisfy those $\boldsymbol{d_n}$ with $\boldsymbol{i < L_n \leqslant j}$** including

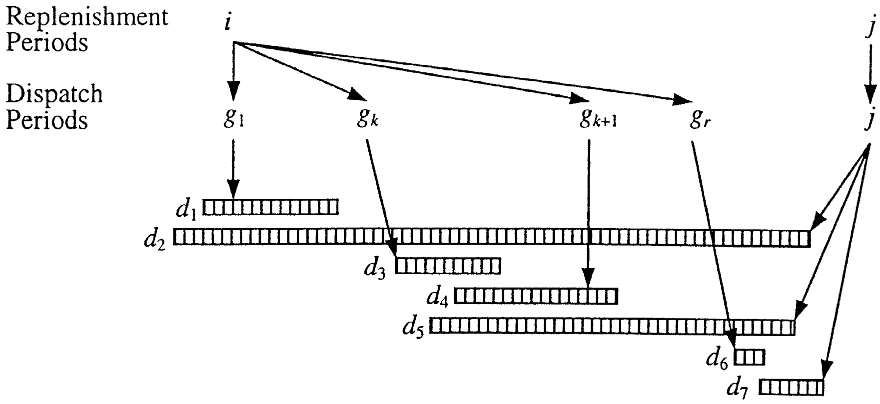

Figure 2. Optimal policy for finding the value of $P(i,j)$.

the set-up costs of replenishments in periods i and j, the procurement costs, the set-up costs of the dispatches during periods i and j, the TPW holding costs, and the DC pre-shipping penalties. For $1 \leqslant t \leqslant T+1$, define **$J(t)$ as the minimum total cost of satisfying those d_n with $1 \leqslant L_n \leqslant t$ where t is a replenishment period**. Recall that we assume $K_0 = K_{T+1} = S_0 = S_{T+1} = 0$ and there is no demand d_n with $L_n = T+1$. Thus, **the optimal cost is $J(T+1)$**. It follows that this problem can be solved using the following recursive equations:

$$J(0) = 0,$$

$$J(j) = \min\{J(i) + P(i,j) - K_i - S_i : 0 \leqslant i < j\}, \quad \text{for } j = 1, \ldots, T+1. \tag{17}$$

Observe that the values of $J(i)$ and $P(i,j)$ include, by definition, the set-up costs for both replenishment and dispatch in period i. Thus, in order to obtain an expression of $J(j)$, K_i and S_i are subtracted from $J(i) + P(i,j)$. The following lemma is useful for computing the values of $P(i,j)$.

LEMMA 4. *There exists an optimal solution such that, if i and j are consecutive replenishment periods, and $g_1 < g_2 < \cdots < g_r$ are successive dispatch periods supplied by the replenishment in period i, where $g_1 = i$ and $g_r \leqslant j$, then the following observations are true for d_n with $i < L_n \leqslant j$ (see the illustration in Figure 2).*

(*i*) *If $E_n \leqslant g_1$, then d_n is satisfied by*

- *the dispatch in period $g_1 = i$, if $p_i \leqslant p_j + w(L_n, j-1)$,*
- *the dispatch in period j, if $p_i > p_j + w(L_n, j-1)$.*

(*ii*) *If* $g_k < E_n \leqslant g_{k+1}$ *for some* k, $1 \leqslant k < r$, *then* d_n *is satisfied by the period, depending on which of the following is minimum*:

- $p_i + h'(i, g_k - 1) + h(g_k, E_n - 1)$,
  (*satisfied by the dispatch in period* g_k),
- $p_i + h'(i, g_{k+1} - 1) + w(L_n, g_{k+1} - 1)$,
  (*satisfied by the dispatch in period* g_{k+1}),
- $p_j + w(L_n, j - 1)$,
  (*satisfied by the dispatch in period* j).

(*iii*) *If* $E_n > g_r$, *then* d_n *is satisfied by*

- *the dispatch in period* g_r, *if* $p_i + h'(i, g_r - 1) + h(g_r, E_n - 1) \leqslant p_j + w(L_n, j - 1)$,
- *the dispatch in period* j, *if* $p_i + h'(i, g_r - 1) + h(g_r, E_n - 1) > p_j + w(L_n, j - 1)$.

Proof. The proof is straightforward; thus, it is omitted. □

Suppose that i and j are consecutive replenishment periods. If all dispatch periods supplied by the replenishment in period i are known, Lemma 4 clearly shows that it is straightforward to compute the value of $P(i, j)$.

For $0 \leqslant i \leqslant u < v < j \leqslant T + 1$, where periods i and j are consecutive replenishment periods, the following is defined:

- $V_{(i,j)}(u, v)$ denotes the minimum cost associated with the *procurement costs*, the *holding costs*, the *pre-shipping penalties*, and the *waiting costs* to satisfy all d_n with $u < E_n \leqslant v$ and $i < L_n \leqslant j$ by the dispatches in periods u, v, and j, where u and v are consecutive dispatch periods supplied by the replenishment in period i.
- $V'_{(i,j)}(0, i)$ denotes the minimum cost associated with the *procurement costs* and the *waiting costs* to satisfy all d_n with $E_n \leqslant i < L_n \leqslant j$ by the dispatches in periods i and j.
- $V''_{(i,j)}(v, j)$ denotes the minimum cost associated with the *procurement costs*, the *holding costs*, the *pre-shipping penalties*, and the *waiting costs* to satisfy all d_n with $i \leqslant v < E_n \leqslant L_n \leqslant j$ by the dispatches in periods v and j, where v is the last dispatch period supplied by the replenishment in period i.

Note that we intentionally do not include the set-up costs of replenishments and dispatches in the above notation.

Define $Q_{(i,j)}(v)$, $0 \leqslant i \leqslant v < j \leqslant T + 1$, as the minimum total cost of satisfying all d_n with $i < L_n \leqslant j$ and $E_n \leqslant v$, where i and j are the consecutive replenishment periods, and v is a dispatch period. The following equations are used to compute the values of $P(i, j)$, for $0 \leqslant i < j \leqslant T + 1$.

For given i and j, such that $0 \leqslant i < j \leqslant T+1$, let

$$Q_{(i,j)}(i) = K_i + S_i + K_j + S_j + V'_{(i,j)}(0,i),$$

and for $v = i+1, i+2, \ldots, j-1$,

$$Q_{(i,j)}(v) = \min\{Q_{(i,j)}(u) + S_v + V_{(i,j)}(u,v) : i \leqslant u < v\}. \tag{18}$$

Thus,

$$P(i,j) = \min\{Q_{(i,j)}(v) + V''_{(i,j)}(v,j) : i \leqslant v < j\}. \tag{19}$$

If all values of $V_{(i,j)}(u,v), V'_{(i,j)}(0,i)$, and $V''_{(i,j)}(v,j)$ have been computed, it takes $O(T^4)$ time to find the values of $P(i,j)$ for all $1 \leqslant i < j \leqslant T$. Next, we show how to find the values of $V_{(i,j)}(u,v), V'_{(i,j)}(0,i)$, and $V''_{(i,j)}(v,j)$.

5.1. COMPUTING $V_{(i,j)}(u,v)$

We need to consider assigning d_n with $i < L_n \leqslant j$ and $u < E_n \leqslant v$ to dispatch in periods u, v, or j, depending on which is the most cost effective, where i and j are two consecutive replenishment periods, and u and v are two consecutive dispatch periods supplied by the replenishment in period i. Let

- $SU(k,i,u) = p_i + h'(i,u-1) + h(u,k-1)$,
- $SV(t,i,v) = p_i + h'(i,v-1) + w(t,v-1)$, and
- $SJ(t,j) = p_j + w(t,j-1)$.

Also, let us define the following notation:

- For $0 \leqslant i \leqslant u < k \leqslant v \leqslant T+1$, let

$$\Delta_1(k,i,u,v) = \max\left\{\begin{array}{l} k-1 \\ t : k \leqslant t \leqslant T \text{ and } SU(k,i,u) \leqslant SV(t,i,v) \end{array}\right\} \tag{20}$$

  For any $t \leqslant \Delta_1(k,i,u,v)$, the cost of satisfying $D(k,t)$ by the dispatch in period u is less than or equal to the cost of satisfying it by the dispatch in period v. It can be shown that $\Delta_1(k,i,u,v) \geqslant \Delta_1(k+1,i,u,v)$ and $k-1 \leqslant \Delta_1(k,i,u,v) \leqslant T$. Thus, the values of $\Delta_1(k,i,u,v)$ for all $0 \leqslant i \leqslant u < k \leqslant v \leqslant T+1$ can be computed in $O(T^4)$ time.
- For $0 \leqslant i \leqslant u \leqslant k \leqslant j \leqslant T+1$, let

$$\Delta_2(k,i,u,j) = \max\left\{\begin{array}{l} k-1 \\ t : k \leqslant t \leqslant j \text{ and } SU(k,i,u) \leqslant SJ(t,j) \end{array}\right\} \tag{21}$$

  For any $t \leqslant \Delta_2(k,i,u,j)$, the cost of satisfying $D(k,t)$ by the dispatch in period u is less than or equal to the cost of satisfying it by the dispatch

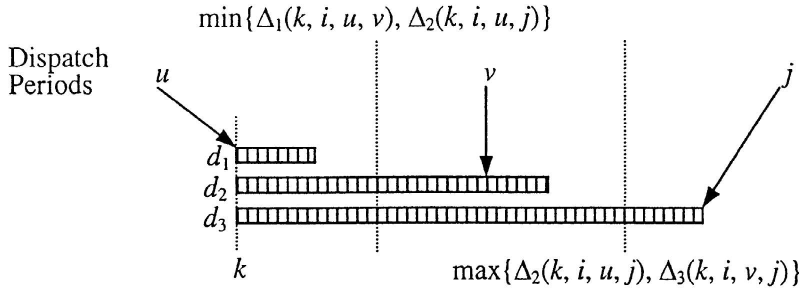

Figure 3. Dispatch periods for $D(k,t), k \leqslant t \leqslant j$.

in period j. It can be shown that $\Delta_2(k,i,u,j) \geqslant \Delta_2(k+1,i,u,j)$ and $k-1 \leqslant \Delta_2(k,i,u,j) \leqslant j$. Thus, the values of $\Delta_2(k,i,u,j)$ for all $0 \leqslant i \leqslant u < k \leqslant j \leqslant T+1$ can be computed in $O(T^4)$ time.

- For $0 \leqslant i \leqslant k \leqslant v < j \leqslant T+1$, let

$$\Delta_3(k,i,v,j) = \max\left\{\begin{array}{l} k-1 \\ t: k \leqslant t \leqslant j \text{ and } SV(t,i,v) \leqslant SJ(t,j) \end{array}\right\} \tag{22}$$

For any $t \leqslant \Delta_3(k,i,v,j)$, the cost of satisfying $D(k,t)$ by the dispatch in period v is less than or equal to the cost of satisfying it by the dispatch in period j. It can be shown that $\Delta_3(k,i,v,j) \leqslant \Delta_3(k+1,i,v,j)$ and $k-1 \leqslant \Delta_3(k,i,v,j) \leqslant j$. Thus, the values of $\Delta_3(k,i,v,j)$ for all $0 \leqslant i < k \leqslant v < j \leqslant T+1$ can be computed in $O(T^4)$ time.

Suppose i and j are two consecutive replenishment periods. Also suppose that u and v, $i \leqslant u < v < j$, are two consecutive dispatch periods supplied by the replenishment in period i. For $u < k \leqslant v$ and $k \leqslant t \leqslant j$, the most cost economical dispatch period to satisfy $D(k,t)$ can be determined by comparing the values of $SU(k,i,u)$, $SV(t,i,v)$ and $SJ(t,j)$. It can be shown that $SU(k,i,u)$ is the minimum of these three if, and only if, $t \leqslant \min\{\Delta_1(k,i,u,v), \Delta_2(k,i,u,j)\}$. Similarly, $SJ(t,j)$ is the minimum if, and only if, $t \geqslant \max\{\Delta_2(k,i,u,j), \Delta_3(k,i,v,j)\} + 1$. It follows that $SV(t,i,v)$ is minimum if, and only if, $\min\{\Delta_1(k,i,u,v), \Delta_2(k,i,u,j)\} + 1 \leqslant t \leqslant \max\{\Delta_2(k,i,u,j), \Delta_3(k,i,v,j)\}$. Thus, for a given $k, u < k \leqslant v$, the following observations are true (see the illustration in Figure 3).

(i) It is most cost effective to satisfy $D(k,t)$ by the dispatch in period u, if

$$k \leqslant t \leqslant \min\{\Delta_1(k,i,u,v), \Delta_2(k,i,u,j)\}.$$

(ii) It is most cost effective to satisfy $D(k,t)$ by the dispatch in period v, if

$$\min\{\Delta_1(k,i,u,v), \Delta_2(k,i,u,j)\} + 1 \leqslant t \leqslant \max\{\Delta_2(k,i,u,j), \Delta_3(k,i,v,j)\}.$$

(iii) It is most cost effective to satisfy $D(k,t)$ by the dispatch in period j, if

$$\max\{\Delta_2(k,i,u,j), \Delta_3(k,i,v,j)\} + 1 \leqslant t \leqslant j.$$

Let us define additional notation that is useful in computing the values of $P(i,j)$ efficiently. Let $B(k,u,v,t)$ be the waiting cost of satisfying all of those d_n with $E_n=k$ and $u \leqslant L_n \leqslant v$ by the dispatch in period t. That is,

$$B(k,u,v,t)=\sum_{j=u}^{v}[w(j,t-1)\cdot D(k,j)].$$

Note that $B(k,u,v,t)=0$ if $u \geqslant v$ and $B(k,u,v,t)=B(k,u,t-1,t)$ if $v \geqslant t$ It can be shown easily that all values of $B(k,u,v,t)$ can be computed in $O(T^4)$ time. Hence, the value of $V_{(i,j)}(u,v)$ is given by

$$V_{(i,j)}(u,v) = \sum_{k=u+1}^{v}\left[\begin{array}{l}\sum_{t=k}^{a(k)}[(p_i+h'(i,u-1)+h(u,k-1))\cdot D(k,t)] \\ +\sum_{t=a(k)+1}^{b(k)}[(p_i+h'(i,v-1)+w(t,v-1))\cdot D(k,t)] \\ +\sum_{t=b(k)+1}^{j}\big[\big(p_j+w(t,j-1)\big)\cdot D(k,t)\big]\end{array}\right]$$

$$= \sum_{k=u+1}^{v}\left[\begin{array}{l}(p_i+h'(i,u-1)+h(u,k-1))\cdot A(k,k,a(k)) \\ +(p_i+h'(i,v-1))\cdot A(k,a(k)+1,b(k)) \\ \quad +B(k,a(k)+1,b(k),v) \\ +p_j\cdot A(k,b(k)+1,j)+B(k,b(k)+1,j,j)\end{array}\right] \tag{23}$$

where $a(k)=\min\{\Delta_1(k,i,u,v),\Delta_2(k,i,u,j)\}$, and
$b(k)=\max\{\Delta_2(k,i,u,j),\Delta_3(k,i,v,j)\}$.

5.2. COMPUTING $V'_{(i,j)}(0,i)$

We need to consider assigning d_n with $1 \leqslant E_n \leqslant i < L_n \leqslant j$ to dispatch in periods i or j, where i and j are consecutive replenishment periods, depending on which is more cost effective. For $1 \leqslant k \leqslant i < t \leqslant j$, $D(k,t)$ is satisfied by the dispatch in period i, if $p_i \leqslant p_j+w(t,j-1)$ (or equivalently, if $t \leqslant \Delta_2(i,i,i,j)$); otherwise, if $p_i > p_j+w(j,j-1)$ (or equivalently, if $t \geqslant \Delta_2(i,i,i,j)+1$), then $D(k,t)$ is satisfied by the dispatch in period j. Hence,

$$\begin{aligned}V'_{(i,j)}(0,i) &= \sum_{k=1}^{i}\left[\sum_{t=i+1}^{a} p_i\cdot D(k,t)+\sum_{t=a+1}^{j}\big(p_j+w(t,j-1)\big)\cdot D(k,t)\right] \\ &= \sum_{k=1}^{i}\big[p_i\cdot A(k,i+1,a)+p_j\cdot A(k,a+1,j)+B(k,a+1,j,j)\big]\end{aligned} \tag{24}$$

where $a = \Delta_2(i, i, i, j)$.

5.3. COMPUTING $V''_{(i,j)}(v, j)$

We need to consider assigning d_n with $i \leqslant v < E_n \leqslant L_n \leqslant j$ to dispatch in periods v or j, depending on which is more cost effective, where i and j are consecutive replenishment periods and v is the last dispatch supplied by the replenishment in period i. For $v < k \leqslant t \leqslant j$, $D(k, t)$ is satisfied by the dispatch in period v, if $p_i + h'(i, v-1) + h(v, k-1) \leqslant p_j + w(t, j-1)$ (or equivalently, if $t \leqslant \Delta_2(k, i, v, j)$); otherwise, if $p_i + h'(i, v-1) + h(v, k-1) > p_j + w(t, j-1)$ (or equivalently, if $t \geqslant \Delta_2(k, i, v, j) + 1$), then $D(k, t)$ is satisfied by the dispatch in period j. Hence, we have

$$\begin{aligned} V''_{(i,j)}(v, j) &= \sum_{k=v+1}^{j} \left[\begin{array}{l} \sum_{t=k}^{c(k)} \big[(p_i + h'(i, v-1) + h(v, k-1)) \cdot D(k, t) \\ \quad + \sum_{t=c(k)+1}^{j} \big(p_j + w(t, j-1) \big) \cdot D(k, t) \big] \end{array} \right] \\ &= \sum_{k=v+1}^{j} \left[\begin{array}{l} (p_i + h'(i, v-1) + h(v, k-1)) \cdot A(k, k, c(k)) \\ + p_j \cdot A(k, c(k)+1, j) + B(k, c(k)+1, j, j) \end{array} \right] \end{aligned} \tag{25}$$

where $c(k) = \Delta_2(k, i, v, j)$.

Clearly, it takes $O(T^5)$ time to compute the values of $V_{(i,j)}(u, v)$ for all $1 \leqslant i \leqslant u < v < j \leqslant T$. It takes $O(T^3)$ time to compute the values of $V'_{(i,j)}(0, i)$ for all $1 \leqslant i < j \leqslant T$; whereas it takes $O(T^4)$ time to compute the values of $V''_{(i,j)}(v, j)$ for all $1 \leqslant i \leqslant v < j \leqslant T$. Finding all values of $P(i, j)$ takes $O(T^4)$ time. The optimal cost, $J(T)$, can be computed in $O(T^2)$ time. Hence, we conclude that the total computational complexity of the case where backlogging is allowed is $O(T^5)$.

6. Conclusions and future research extensions

This paper considers a two-echelon dynamic lot-sizing problem where demands should be satisfied during pre-specified time windows. The problem is challenging because, under demand time window considerations, the structural properties of the underlying dynamic problem are not straightforward to characterize. Unlike the case where time window considerations are not applicable, a policy in which each replenishment/dispatch satisfies demand with consecutive indices is not necessarily optimal. Hence, the paper presents a novel decomposition strategy that leads to the development of efficient algorithms that run in polynomial time.

By laying the groundwork for multi-echelon problems with time window considerations, the paper provides a basis for future work in the area. Our future research includes extensions of the problem to include production capacity constraints, inventory capacity constraints, and cargo capacity constraints. This class of problems is an emerging area of interest due to its applications in the context of supply/transportation contracts and supply chain partnerships. Although the emphasis of this paper is on algorithmic development and efficiency, the methodology presented is useful for analyzing different scenarios of the problem under various possible time windows at the contract design stage. For example, the optimal algorithm developed can be used for quantifying the value of demand time windows, i.e., the costs and benefits due to enlarging the time windows. A numerical investigation of this type requires demand and cost data that are typically available at the warehouse/supplier.

Acknowledgements

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